A Comparison of Software Packages for Verified Linear Programming

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Outline

1. Introduction
2. Numerical Experience
   - Solvers
   - Test environment
   - Results
3. Summary

Why linear programming?

- Most basic form of optimization $\Rightarrow$ simple theory
- Wide range of applications:
  oil refinery problems, flap settings on aircraft, industrial production and allocation, image restoration, linearization, linear relaxations in global optimization, ...

Lovasz, 1980

If one would take statistics about which mathematical problem is using up most of the computer time in the world, then (not including database handling problems like sorting and searching) the answer would probably be linear programming.

- Ordóñez and Freund, 2003:
  71% of Netlib LP problems are ill-posed

Ben-Tal and Nemirovski, 2000

In real-world applications of Linear Programming one cannot ignore the possibility that a small uncertainty in the data (intrinsic for most real-world LP programs) can make the usual optimal solution of the problem completely meaningless from a practical viewpoint.

Definition (Linear Program (LP))

Find the optimal value f^* of a linear objective function $c^T x$ subject to

- linear constraints $Ax \leq a, \; Bx = b$ and
- simple bounds $\underline{x} \leq x \leq \overline{x}$.

- Simple bounds may be infinite

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Constraint satisfaction software to solve LP rigorously

ICOS (Lebbah), RealPaver (Granvilliers)

- No optimization
  $\Rightarrow$ lower bound from non-satisfiable problems
- Naturally fit for checking feasibility, not for unboundedness
- ICOS: constraint programming, interval analysis
- RealPaver: branch-and-prune, interval Newton method

Rational arithmetic software to solve LP rigorously

perPlex (Koch), exlp (Kiyomi), QSopt_ex (Applegate et. al)

- Rational arithmetic $\Rightarrow$ requires rational solution
- perPlex proof of concept
  - Rational verification of approximate solution, does not compute solution
  - Does not verify infeasibility, unboundedness
- exlp
  - Rational simplex
  - Verifies infeasibility and unboundedness
- QSopt_ex
  - Multiprecision simplex with rational verification
  - Verifies infeasibility

Global optimization software to solve LP rigorously

GlobSol (Kearfott), Numerica (Hentenryck), COSY (Berz)

- Verify infeasibility, not unboundedness
- GlobSol: interval branch-and-bound, automatic differentiation, constraint propagation, interval Newton, ...
- Numerica: interval methods (Hansen–Sengupta), constraint satisfaction
- COSY: branch-and-bound, Taylor model arithmetic

Global optimization software to solve LP rigorously

GlobSol (Kearfott), Numerica (Hentenryck)

- Verify infeasibility, not unboundedness
- GlobSol: interval branch-and-bound, automatic differentiation, constraint propagation, interval Newton, ...
- Numerica: interval methods (Hansen–Sengupta), constraint satisfaction

Global optimization software to solve LP rigorously

GlobSol (Kearfott)

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- GlobSol: interval branch-and-bound, automatic differentiation, constraint propagation, interval Newton, ...

LP software to solve LP rigorously

Lurupa (Keil)

- Algorithms developed by Jansson at IRC
- Iteratively enclose near optimal feasible points
- Certificates for infeasibility and unboundedness
- Verified condition numbers

Are we comparing apples and oranges?

What is solving?

- Return a rigorous result for the LP
- Type of result (exact, enclosure of optimal, near optimal point) often secondary from application point of view
- Lower bound (ICOS, RealPaver) important for rigorous branch and bound schemes

Packages for different tasks with different outputs – fair?

- All can solve LP $\Rightarrow$ look at performance
- Test whether exploiting structure is necessary
- LP is an easy dimension test,
  general problems of same dimension are much harder

- 2.2 GHz Intel Xeon with 3 GByte available to the process
- SUSE Linux 10.0
- All packages compiled with unmodified gcc 4.1.2

- 27 random problems after Rosen and Suzuki
  - Choose solution ⇒ generate problem from KKT
  - Non–degenerate, exactly known, integral solution
  - 5–1500 variables
  - 3 instances per dimension set
    (n variables, n inequalities, $0.5n$ equations)
- 103 feasible real-world problems from Netlib and Meszaros's collection
  - Less than 1500 variables
  - Large finite search region
- 28 infeasible real-world problems from Meszaros's collection
- 1 hour timeout

solCheck

- Uses rational arithmetic (GMP)
- Checks rational solution for feasibility
  (exlp, QSopt_ex, perPlex)
- Checks if enclosure contains feasible point
  (ICOS, RealPaver, GlobSol, Lurupa)

cmpSol

- Uses rational arithmetic (GMP)
- Compares optimal values returned by rational solvers
  (exlp, QSopt_ex, perPlex)
- Checks if enclosure contains
  optimal value returned by rational solvers
  (GlobSol, Lurupa)

ICOS

- Lower bounds with 12 correct digits

RealPaver

- Two lower bounds with 8 correct digits
- One lower bound with 4 correct digits

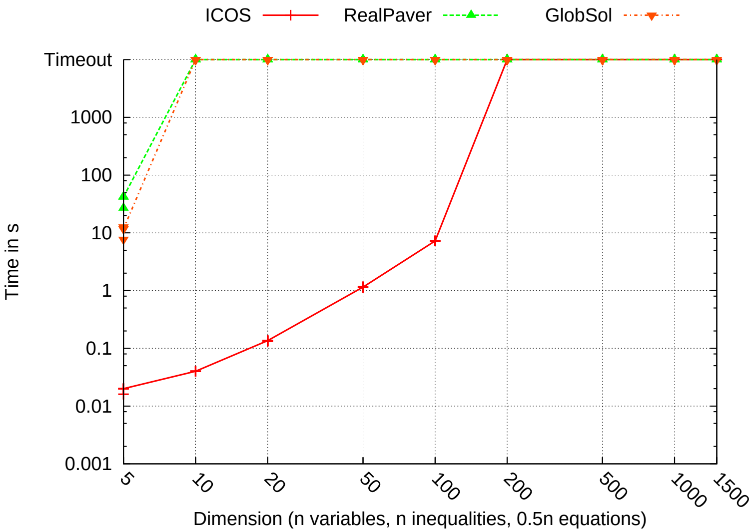
GlobSol

- Enclosures with 9 – 13 guaranteed digits

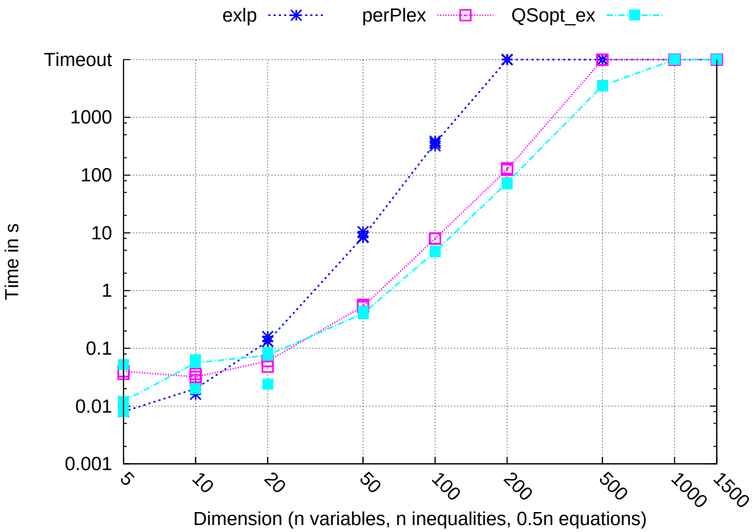
Lurupa

- Enclosures with median 8 guaranteed digits
- One enclosure with 4 guaranteed digits

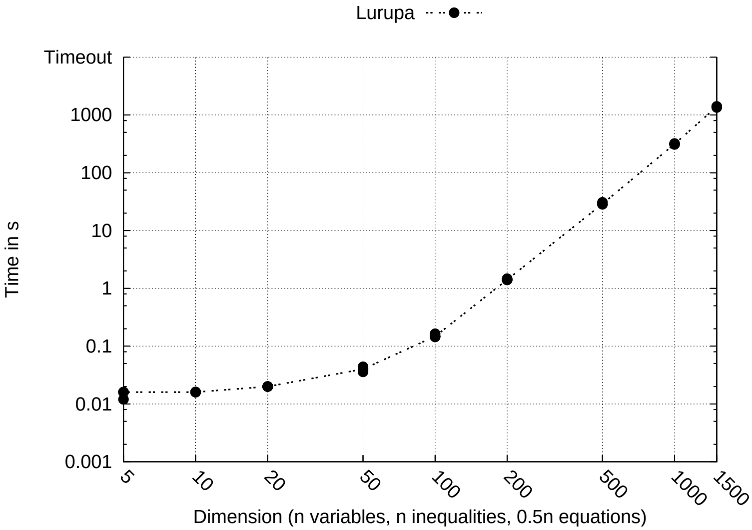
Running times for random problems



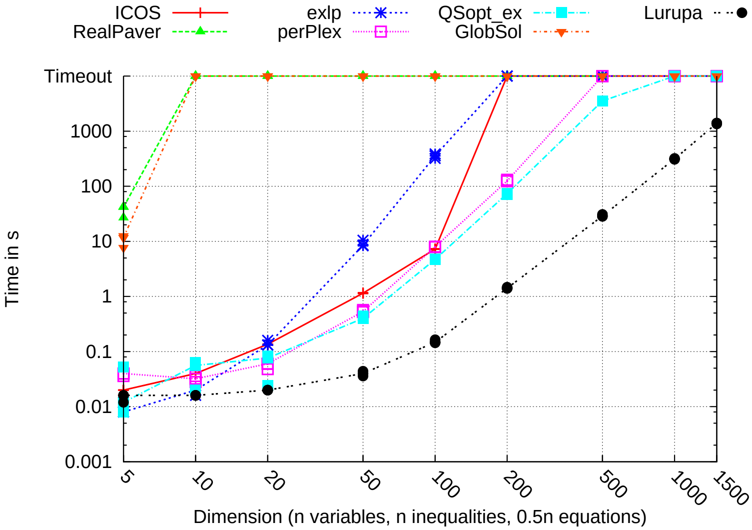
Running times for random problems



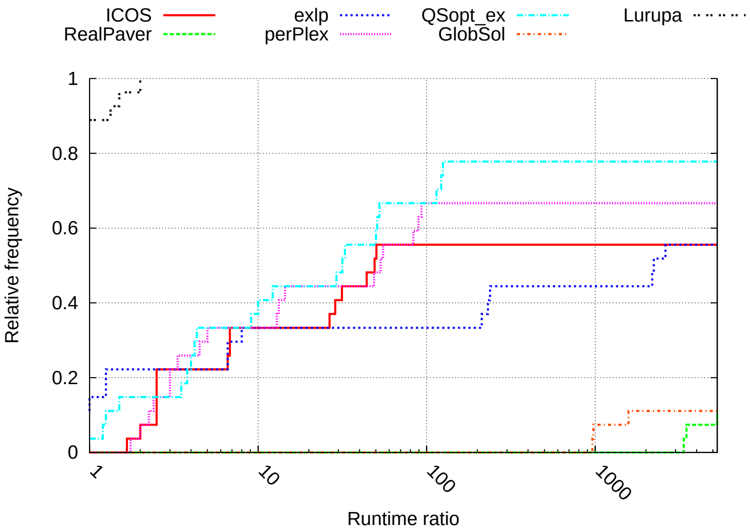
Running times for random problems



Running times for random problems



Performance profile on random problems



Results for feasible real-world problems

ICOS

- Lower bounds with 12 correct digits
- Runs out of memory when failing
- Claims three problems to be infeasible

RealPaver

- Lower bounds with 12 correct digits
- One lower bound with 8 correct digits

exlp

- 7 suboptimal solutions (fixed by disabling preprocessing)
- 6 infeasible solutions
- 1 LP file claimed to be malformed

More results for feasible real-world problems

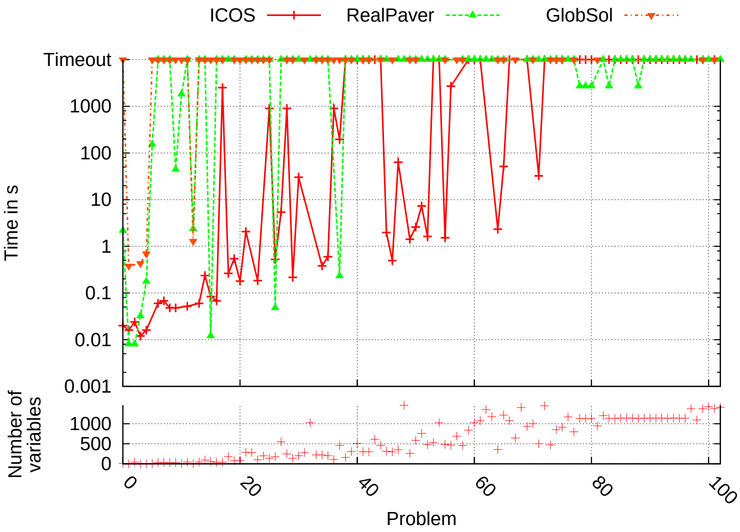
GlobSol

- 4 enclosures with $\approx$ 15 guaranteed digits
- 41 problems not representable due to Fortran line length limitation
- 7 problems claimed to be infeasible

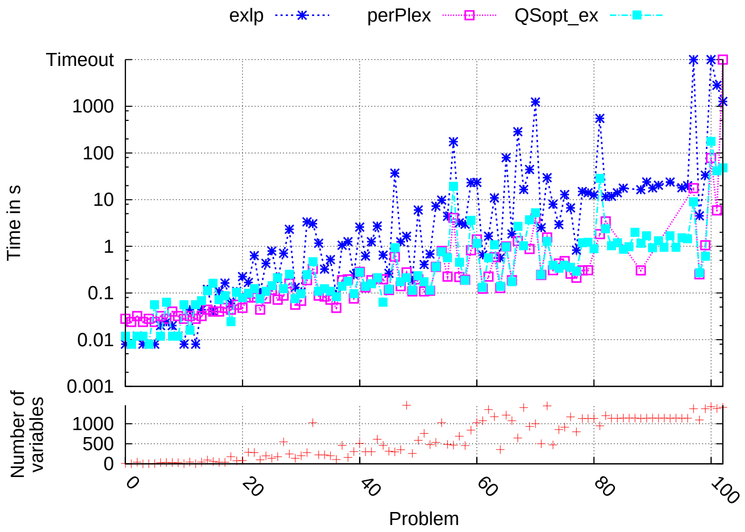
Lurupa

- 59 enclosures with median 8 guaranteed digits
- One enclosure with no guaranteed digit
  (lower bound has 7 common digits with approximate optimal value)
- 34 problems without upper bound
  lower bound in median 8 common digits with approximate optimal value

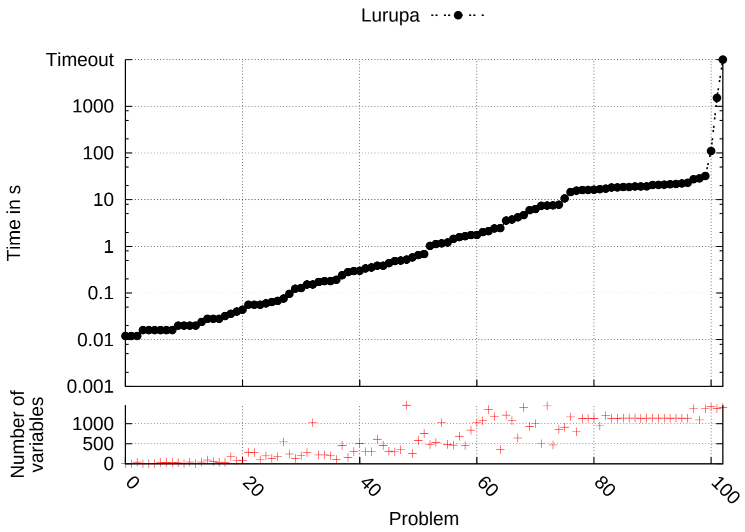
Running times for feasible real-world problems



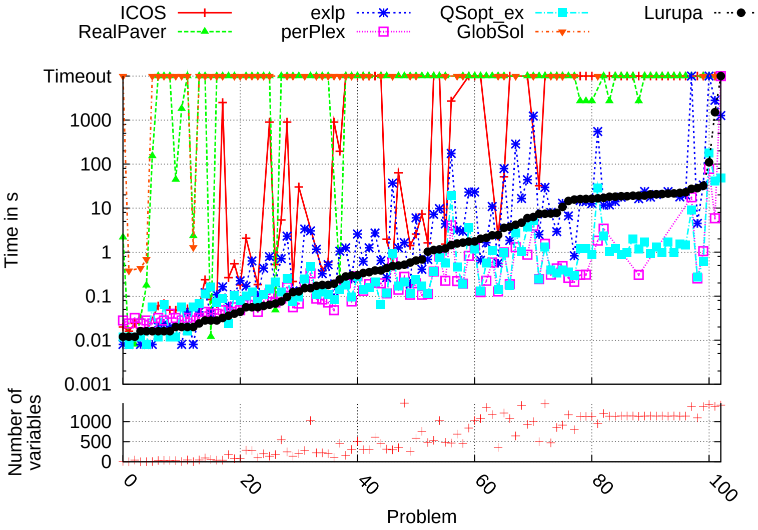
Running times for feasible real-world problems



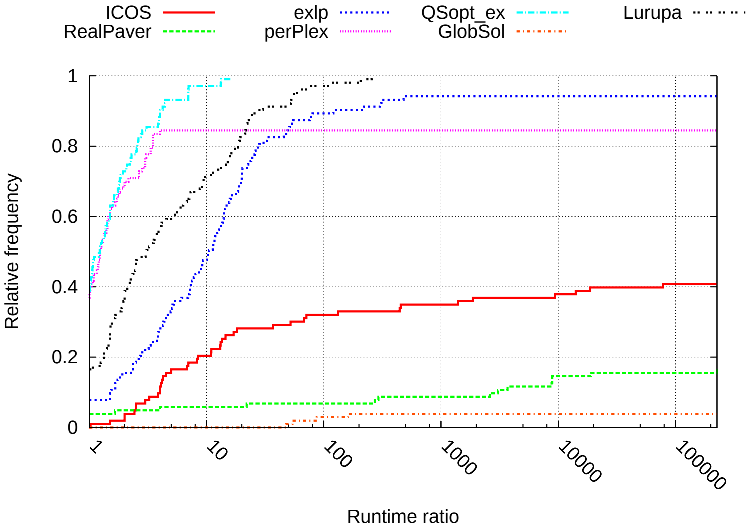
Running times for feasible real-world problems



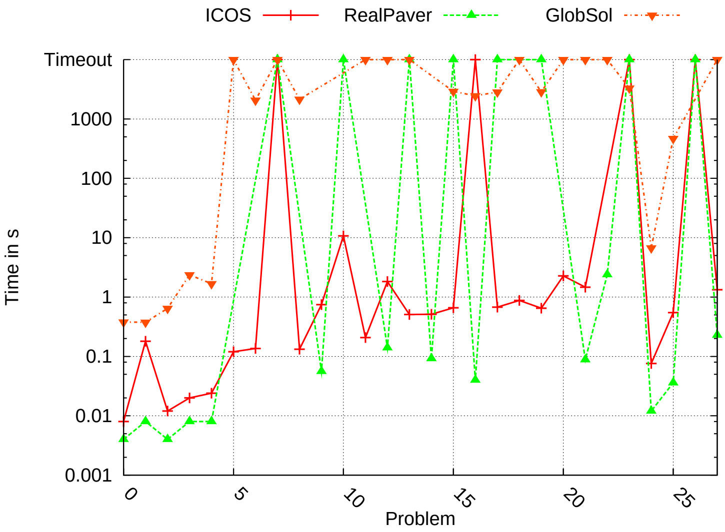
Running times for feasible real-world problems



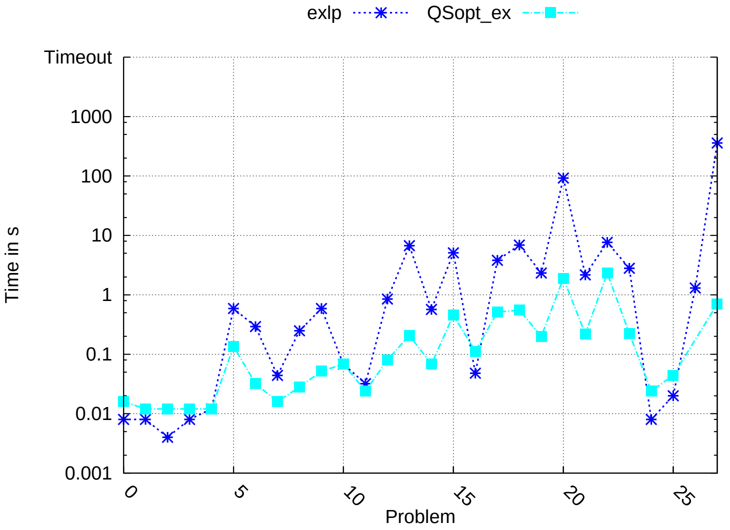
Performance profile on feasible real-world problems



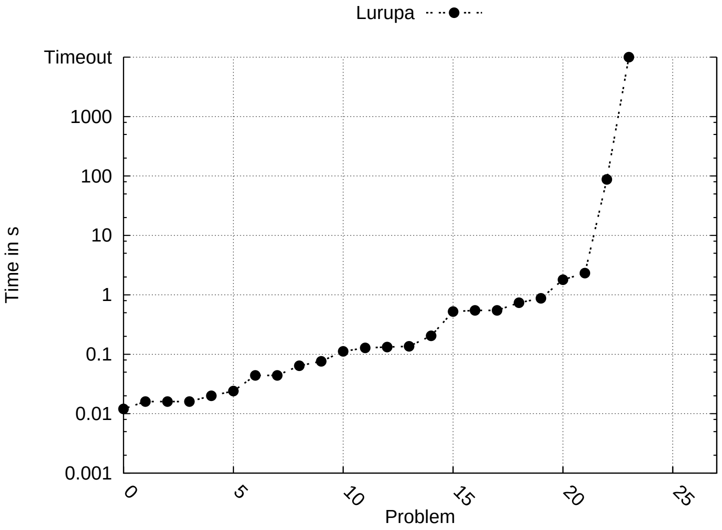
Running times for infeasible real-world problems



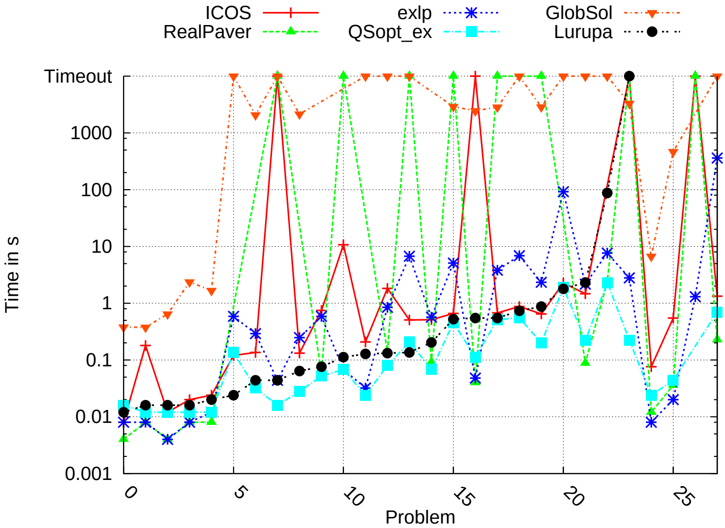
Running times for infeasible real-world problems



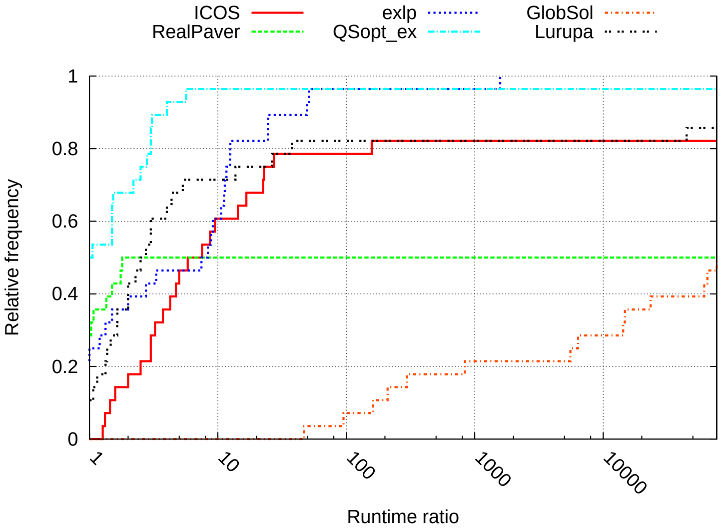
Running times for infeasible real-world problems



Running times for infeasible real-world problems



Performance profile on infeasible real-world problems



Outline

Summary

- Rigorous global optimization software can verify solutions for small problems
  (ICOS hundreds of variables,
  RealPaver and Globsol 20 – 30 variables)
- Larger dimensions require to exploit special structure
- Lurupa implements appropriate algorithms and solves problems with $\approx$ 10000 variables and constraints in reasonable time

Thank you for your

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