Value-Restricted Functions for Robust and Simultaneous Stability

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Different problems, like robust stability of interval plants, stability dependent of delay, and simultaneous stabilizability of three systems are treated via a single method. The value set properties for certain derived functions allow usage of computational analytic inequalities. This approach leads to necessary conditions for the parameters of the stability problems.

1 Introduction

Consider the following polynomial with coefficients in intervals.

$$p(z) = \sum_{j=0}^{n} a_j z^j, \quad \text{where} \quad a_j \in \left[a_j^-, a_j^+\right] \subset \mathbb{R}. \tag{1}$$

What is the largest possible diameter of the coefficient intervals? Consider likewise the case of real Hurwitz stable polynomials. While it is known that four coefficients delimit stability, we might ask whether there is a condition relating three coefficients. Moreover, we might ask if there are upper and lower limits to a coefficient depending on three coefficients only? Consider the problem of bistable stabilization of a single plant where the zeros and poles lie in a small circle around the origin. Is there a *quantitative* limit to stabilizability?

To study these questions, we propose the following general scheme. Define a function dependent on a single parameter such that the auxillary function has a 'nice' mapping property. Give an analytic characterization of the mapping property, and obtain computable bounds there-of. This strategy leads to results which improve and/or supplement exisiting results in the named areas, and which eventually are best possible.

2 Derived functions and mapping properties

Consider the Schur-stable interval polynomial (1). To study the diameter of a_k consider the meromorphic function

$$f_k(z) \quad := \quad \frac{(a_{n-k}^+ - a_{n-k}^-) \cdot z^k}{a_n^- + \sum a_{n-j}^- z^j} \notin (-\infty, -1] \quad \forall z.$$

The function maps the unit disc to the slit half-plane $\mathbb{C} \setminus (-\infty, -1]$. Applying an appropriate conformal mapping, we obtain a bounded mapping from the unit disc to itself. It is well-known that the coefficients of a bounded holomorphic function are bounded. One quantitative expression of this fact is the following result.

Proposition 2.1 (Gutzmer)
Given $g(z) = \sum_{v=0}^{\infty} g_v z^v$ *with* $|g(z)| < 1 \, \forall z \in \mathbb{D}$. *Then*

$$\sum_v |g_v|^2 \leq 1 \ \forall v.$$

We arrive at the following theorem [2].

Theorem 2.2 *Given* $n+1$ *real intervals* $\left[a_j^-, a_j^+\right], \ j = 0, \dots n$, *with* $a_n^- > 0$. *Consider* $p(z) = \sum_{j=0}^{n} a_j z^j$ *with* $a_j \in \left[a_j^-, a_j^+\right]$. *If the interval family is Schur-stable, then*

$$\left|a_l^+ - a_l^-\right| \leq 4 \cdot \left|a_n^-\right|, \ l = 0, \cdots, n-1.$$

The constant 4 *is sharp.*

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The result may be extended to multi-variate polynomials in the presence of what is called a principal term in the theory of quasi-polynomials, see [1].

Consider the case of real Hurwitz-stable polynomials $P(z) = U(z) + iV(z)$, where U and V are real on the real axis. According to the Hermite-Biehler theorem Hurwitz-stable polynomials have real and imaginary parts with roots only on the imaginary axis. This allows to define the following function

$$\frac{U'(z)}{U(z)} \quad \text{and} \quad \frac{V'(z)}{V(z)} \quad \text{for} \quad \Re z > 0. \tag{2}$$

The explicit form of the logarithmic derivative allows to conclude that the functions in (2) map the right half-plane to itself. Functions with this mapping property may be expressed via certain Stieltjes integrals. In the case of an odd, real function bounded near zero we use the following auxillary result.

Corollary 2.3 *A positive analytic function $f(z)$ has an asymptotic expansion near the origin given by*

$$f(z) \sim c_0 z + c_1 z^3 + c_2 z^5 + O(z^7), z \to 0, \Re z > 0,$$

only if all determinantal expressions

$$H_k^{(0)} := |(c_{i+j})_{\substack{i=0,\ldots,k\\ j=0,\ldots,k}}|, H_k^{(1)} := (-1)^{k+1}|(c_{i+j+1})_{\substack{i=0,\ldots,k\\ j=0,\ldots,k}}| \tag{3}$$

are non-negative.

This allows to conclude the following which supplements and improves the lower bound in [6].

Theorem 2.4 *Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be Hurwitz stable with positive coefficients. Designate by N the integer $\lfloor n/2 \rfloor$. Let $\tau \in \{0, 1\}$. Then the following holds true for all p such that $1 \leq p < N - 1$.*

$$a_{2p+\tau} \ \geq \ \sigma_p \cdot \sqrt{a_{2p-2+\tau} a_{2p+2+\tau}},$$

where $\sigma_p := \sqrt{\frac{p+1}{p} \frac{N-(p-1)}{N-p}}$.

If $a_{2p+\tau} a_{2p+2+\tau} = f \cdot (\sigma_p \cdot \sigma_{p+1})^2 a_{2p-2+\tau} a_{2p+4+\tau}$ where $f \geq 1$ obtain the following constraint for the coefficients.

$$U(f) \cdot \sigma_p \cdot \sqrt{a_{2p-2+\tau} a_{2p+2+\tau}} \geq \quad a_{2p+\tau} \quad \geq L(f) \cdot \sigma_p \cdot \sqrt{a_{2p-2+\tau} a_{2p+2+\tau}},$$

where

$$U(f) := \sqrt{3} \cdot \sqrt{\left(9\,f^2 - 1 + 10\,f + \sqrt{81\,f^4 + 82\,f^2 - 108\,f^3 + 1 - 20\,f}\right)/(36f)} \quad \geq \sqrt{2},$$

$$L(f) := \sqrt{3} \cdot \sqrt{\left(9\,f^2 - 1 + 10\,f - \sqrt{81\,f^4 + 82\,f^2 - 108\,f^3 + 1 - 20\,f}\right)/(36f)} \quad \geq 1.$$

This result generalizes to time-delay systems described by quasi-polynomials, and to the largest class of entire functions where the Hermite-Biehler result and Rolle's theorem simultaneously hold true. The precise statement and result will be published elsewhere.

The problem of simultaneous stabilization of three systems may be treated by the above outlined approach as well, see [3] and [5]. Fo illustration, consider one of Blondel's benchmark problems known as 'belgian chocolate problems' after an offering published in [4]. The above approach allows to conclude [3] that the plant $z/(z^2 + \delta^2)$ is not bistable stabilizable for $\delta < 7.5 \cdot 10^{-4}$.

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