SUBORDINATION AND STABILIZATION OF PLANT FAMILIES

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Abstract: We introduce a new tool for the analysis of robust stability and simultaneous stabilizability questions. By a single approach we derive limits for controller parameters in low-order synthesis schemes as well as constraints for unit interpolation problems. Especially, we answer a question by Vijay V. Patel on stabilization of three special systems.

Keywords: Schwarz' Lemma, Subordination, Controller parameter limits, Simultaneous stabilizability, Transcendental problems.

1. MOTIVATION

Schwarz' Lemma in combination with the maximum principle provided Schur, Nevanlinna and Pick with the means to solve the problem of analytic rational interpolation inside the unit disc. After the purely mathematical problem was solved, engineering applications arose from problems of filter design, broadband matching, control system synthesis, see, e.g., (Delsarte *et al.* 1981). It is therefore of interest to pursue generalizations of Schwarz' Lemma to the point of applicability in the area of robust control. We will focus on subordination as developed by Littlewood from Lindelöf's abstraction of Schwarz' result.

We will present facts from Littlewood's theory of subordination (for ref., see also (Nehari 1952)) which will enable us to treat rational systems with unknown coefficients varying over bounded intervals. We derive limits for the absolute value as well as the interval range. In connection with the problem of simultaneous stabilization of three systems, we answer a question by Vijay V. Patel on stabilization of three special systems. The analysis opens the possibility to derive constraints for unit interpolation inside the unit disc.

2. A TOOL OF COMPLEX ANALYSIS

It is well-known that the coefficients of a polynomial

$$p(z) = \sum_{i=0}^{n} a_i z^i = a_n \prod_{j=1}^{n} (z - \zeta_j)$$

may be expressed via the elementary symmetric functions S_i. We have $a_i/a_n = (-1)^{n-i} S_{n-i}(\zeta_1, \zeta_2, \ldots, \zeta_n)$, where

$$S_0(z_1, \ldots, z_n) \equiv 1, S_\nu := \sum_{1 \le j_1 < j_2 < \ldots j_\nu \le n} z_{j_1} \ldots z_{j\nu}.$$

Denote the unit disc in the following by $\mathbb{D}$. Let us consider the case of a polynomial with all roots inside $\mathbb{D}$, i.e. a Schur stable polynomial. Obviously,

$$|a_i/a_n| = |S_i(\zeta_1, \zeta_2, \ldots, \zeta_n)| \le \binom{n}{i}. \qquad (1)$$

Hence, we conclude that if the Schur-stable polynomial is monic, i.e. $a_n = 1$, the polynomial is bounded on $\mathbb{D}$ by 2^n. This does not reflect the full geometrical information available. Consider the reciprocal polynomial

$$p^*(z) = \sum_{i=0}^{n} a_i z^{n-i} = z^n p(1/z) = a_n \prod_{j=1}^{n} (1 - z \cdot \zeta_j)$$

with all roots outside the unit disc. Dieudonné (Dieudonné 1934) considered the geometrical mean of $(1-z\cdot\zeta_1),\ldots,(1-z\cdot\zeta_n)$ to find that this defines a holomorphic function $\phi(\cdot)$ as in the following theorem.

Theorem 1. Given a polynomial $P(z)$ with no roots in the unit circle and $P(0)=1$. Then

$$P(z) = (1-\phi(z))^n$$

for a holomorphic function $\phi(z)$ on $\mathbb{D}$ with $\phi(0)=0$ and $\phi(\mathbb{D})\subset\mathbb{D}$.

The above representation offers the possibility to study the polynomial $P(z)$ as a function which cannot take values outside the range of $(1-z)^n$. Derive bounds to $\phi(z)$ using the following fundamental result.

Lemma 2. (Schwarz) Let f be a holomorphic function on the unit disc bounded there in modulus by unity. Moreover, let $f(0)=0$. Then $|f(z)|\le|z|$.

We find the following upper and lower bounds for the range of $P(z)$ based on the representation in Th. 1.

Corollary 3. Given a polynomial $P(z)$ with no roots in the unit circle and $P(0)=1$. For $z\in\mathbb{D}$ with $|z|=r$ we have that

$$(1-r)^n \le |P(z)| \le (1+r)^n. \qquad (2)$$

The correspondence of functions as in Theorem 1 was studied systematically by Littlewood, cf. (Littlewood 1944), who coined the term *subordination.*

Definition 1. Let $\hat{f}$ be a meromorphic function in $\mathbb{D}$, and ω any function satisfying the prerequisites of Schwarz' Lemma as stated above. If f is any function of the form

$$f(z) = \hat{f}(\omega(z))$$

we shall say that f is *subordinate* to $\hat{f}$.

For a schlicht (i.e. univalent) function $\hat{f}$ this means that f takes all values inside the simply-connected domain $\hat{f}(\mathbb{D})$. Littlewood used the superordinate functions explicitly (Littlewood 1944) to study the following function properties of f subordinate to $\hat{f}$. I. Suppose complete knowledge of the superordinate function $\hat{f}$ is available.

- The image of $\mathbb{D}$ under f lies inside the algebraic surface $\hat{f}(\mathbb{D})$. Especially, if $\hat{f}$ is regular in $|z|=r$ we obtain upper bounds for f in $|z|=r$.
- Assuming that $\hat{f}$ is regular at $z=0$, we obtain upper bounds for the coefficients of $f(z)=\sum a_i z^i$, especially for the expressions $|a_i|$, $\sum_{i=0}^{k}|a_i|^2$, $\sum_0^\infty|a_i|^2r^{2n}$.

II. Moreover, we might use this principle in the opposite direction. Assuming subordination of f to a schlicht $\hat{f}$, the above implications would follow. If we know from a property of f or the class it belongs to, that this is impossible, we may conclude that $f(\mathbb{D})$ does not lie inside $\hat{f}(\mathbb{D})$. This is to say, that f must take values which $\hat{f}$ does not.

One important example for I. is the following classical result((Littlewood 1944) Th.212, Th.215).

Theorem 4. Suppose $f(z)=\sum_{i=0}^{\infty}a_iz^i$ is regular at $z=0$ and subordinate to the regular function $\hat{f}(z)=\sum_{i=0}^{\infty}b_iz^i$. Then

$$|a_1|\le|b_1|, |a_2|\le\max\{|b_1|,|b_2|\},$$
$$\sum_{i=1}^{k}|a_i|^2\le\sum_{i=1}^{k}|b_i|^2.$$

Geometrical properties of $\hat{f}(D)$ have impact on the subordinate function $f(\cdot)$.

Theorem 5. Assum. and Notation as above. If $\hat{f}(z)$ is convex on $\mathbb{D}$, i.e. $\hat{f}$ univalent and $\hat{f}(\mathbb{D})$ convex, then

$$|a_i|\le|b_1|.$$

An important example in direction II. is the following result (Littlewood 1944), Th. 217.

Theorem 6. Let f and F be meromorphic functions with $F(0)=f(0)=0$ and $F'(0)=f'(0)=1$. Let $0<t\le1, 0<\rho\le1$. Then

$$f(\rho z) \text{ is not subordinate to } tF(\rho z),$$

except in the case $t=1$, $f(z)\equiv F(z)$.

Littlewood (Littlewood 1944) calls the

> systematic application of these ideas in both directions (I and II) [...] the "principle of subordination".

3. LIMITING POLYNOMIAL COEFFICIENTS

Bhattacharyya et al. have developed complete parametrization of PID- as well as first-order controllers for discrete-time rational systems (Xu *et al.* 2001), (Tantaris, R.N.; Keel, L.H.; Bhattacharyya, S.P. 2003). From the parametrization, we might choose the optimal controller regarding performance, robustness and especially fragility. The complete set of three parameters describing the controllers is obtained from the solution of a two-variables LP-problem accompanying a sweep for the third parameter over the real line. The extension of their method to interval plants necessitates a two parameter-sweep (Ho *et al.* 1998). We suggest to limit the possible range of parameters using the above results as in the following estimates.

Example:
Consider the stabilization of the system (from (Tantaris, R.N.; Keel, L.H.; Bhattacharyya, S.P. 2003))

$$\frac{24z^5+72z^4+19z^3+81z^2+84z+95}{76z^6+42z^5+56z^4+59z^3+24z^2+z+15}=\frac{N(z)}{D(z)},$$

with generic PID-controller: $K_P + K_I \cdot T\frac{z}{z-1} + \frac{K_D}{T}\frac{z-1}{z}$ reparametrized as

$$\frac{K_2z^2+K_1z+K_0}{z(z-1)}. \tag{3}$$

Compute the characteristic polynomial $P = N(z)(K_2z^2+K_1z+K_0)+D(z)(z(z-1))$, of the closed feedback loop with plant $Num(z)/Denom(z)$ and discrete PID-controller, and normalize to the monic P_1

$$P_1(z)=\frac{5K_0}{4}1+(95K_1+84K_0-15)/76\cdot z+\dots \\ +\frac{72K_2+24K_1+14}{76}z^6+\frac{24K_2-34}{76}z^7+z^8,$$

Estimates (1) from Viète's formulas give $-4/5 \le K_0 \le 4/5$, $-6.95 \le K_1 \le 7.27$ and $-29.66 \le K_2 \le 29.08$. Considering the normalized reciprocal polynomial $P_1^*(z) = z^8 \cdot P_1(1/z)$ we obtain from Th. 1 together with Th.4 the improvement $-23.92 \le K_2 \le 26.75$. If we evaluate $P_1^*(z)$ for $z = 1/100$ we may improve this estimate to $-22.32 \le K_2 \le 26.75$. This yields an improvement of at least 15 percent over the estimates from Viète's formulas, and hence implies a proportionally lighter work-load of function evaluations and LP-solving in the parametrization schemes by Bhattacharyya et al.

Considering the proposed two parameter-sweep which implies checking stability of interval polynomials for each parameter value, it might be useful to determine upper bounds on the interval coefficient range. Consider an interval family of real, Schur-stable polynomials, i.e., the collection of $p(z) = \sum_{j=0}^{n} a_j z^j$, where $a_j \in \left[a_j^-, a_j^+\right] \subset \mathbb{R}$. Assume $a_n^- > 0$ (so that the family is degree-invariant). Suppose every element of the family is Schur-stable. Thus, every family member $p(z)$ has all its roots inside the unit disc. Equivalently, every $p^*(z) := \sum_{j=0}^{n} a_{n-j} z^j = z^n p(1/z)$ has all zeros outside the closed unit disc $\overline{\mathbb{D}}$. Hence, Schur-stability of the family is equivalent to the following condition for all $a_{n-j} \in \left[a_{n-j}^-, a_{n-j}^+\right]$:

$$\forall z \in \overline{\mathbb{D}} : p^*(z) = \sum_{j=0}^{n} a_{n-j} z^j \neq 0. \tag{4}$$

Consider the special family member

$$p^-(z) := a_n^- + \sum_{j=1}^{n} a_{n-j}^- z^j.$$

Define, for all k, $d_k := a_{n-k}^+ - a_{n-k}^-$ (≥ 0). Hence, for $z \neq 0, d_k > 0$ the rational function $p^-(z)/(d_k \cdot z^k)$ takes no values in $[-1, 0]$, as otherwise $p^-(z) + \tau \cdot d_k \cdot z^k = 0$ for some $\tau \in [0, 1]$, contradicting (4).

Switching the role of numerator and denominator, adding unity yields

$$f_k(z) := 1 + \frac{d_k \cdot z^k}{p^-(z)} \notin (-\infty, 0] \quad \forall z : |z| < 1, \\ F_k(z) := \sqrt{f_k} \notin \{z \in \mathbb{C} : \Re z \le 0\} \tag{5}$$

The above function is holomorphic inside $\mathbb{D}$ as the denominator is free from zeros according to (4).

As $(1+z)/(1-z)$ maps the unit disc conformally to the right halfplane, the function $F_k(z) = 1 + \frac{1}{2}\frac{d_k z^k}{2a_n^-} + \dots$ is subordinate to

$$\lambda(z) := \frac{1+z}{1-z} = 1 + 2\sum_{1}^{\infty} z^i. \tag{6}$$

Using the theory of subordination, esp. Theorem 5 we re-derive the result in (Batra 2004).

Theorem 7. Given $n+1$ real intervals $\left[a_j^-, a_j^+\right], j = 0, \dots n$, where $a_n^- > 0$. Suppose every polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ with coefficients $a_j \in \left[a_j^-, a_j^+\right]$ is Schur-stable. Then the following holds true.

$$\left|a_l^+ - a_l^-\right| \le 4 \cdot \left|a_n^-\right|, \; l = 0, \cdots, n-1.$$

While the above result allows to limit the interval coefficient range, we might vary our construction to obtain limits for the volume of parameter boxes describing stabilizing controllers.

Example (continued):
Write $N(z)/D(z)$ as $\sum_{j=0}^{5} a_j z^j / \sum_{k=0}^{6} b_k z^k$.

Consider the reciprocal P^* of the characteristic polynomial of the closed feedback-loop with generic PID (3), and sort the terms dependendent on K_j :

$$P^*(z) = z\sum_{j=0}^{5} z^j K_2 a_{5-j} + z^2 \sum_{j=0}^{5} z^j K_1 a_{5-j} + z^3 \sum_{j=0}^{5} z^j K_0 a_{5-j} + \sum \dots .$$

We may vary our construction, and define the functions

$$\sqrt{1 + \frac{d(K_k)\cdot z^{2-k}(a_5 z + a_4 z^2 + \dots)}{P^*(z)}}, k = 0, 1, 2.$$

These are subordinate to the mapping λ as in (6), and as above we obtain a coefficient box for fixed extra parameters:

$$\max\{d(K_0), d(K_1), d(K_2)\} \leq 4\frac{b_6}{a_5} \overset{Plant}{=} 12.666.$$

The above computation shows more generally that in controller space the largest coefficient ball of stabilizing PID controllers is bounded in absolute terms which delimits specifications regarding fragility. We have the following new result.

Theorem 8. Given a strictly proper discrete-time plant $\frac{\sum_0^{n-1} a_k z^k}{\sum_0^n b_k z^k}$ stabilized in closed-loop by a PID-controller . The l_2-norm of the largest stabilizing PID-coefficient ball is no larger than

$$4 \cdot \sqrt{3} b_n / a_{n-1}.$$

4. SIMULTANEOUSLY STABILIZING THREE SYSTEMS

In 1994, Blondel, Gevers, Mortini and Rupp (Blondel *et al.* 1994) considered stabilizability of the following three systems:

$$P_0 = \frac{2}{17}\frac{s-1}{s+1}, P_1 = \frac{(s-1)^2}{(9s-8)(s+1)}, P_2 = 0. \quad (7)$$

The co-prime factorization approach has led V.V. Patel (Patel 1999) to the solution of a more general problem. Consider the three linear systems with parameter δ:

$$P_0 = 2\delta\frac{s-1}{s+1}, P_2 = 0, \quad (8)$$
$$P_1 = \frac{2\delta(s-1)^2}{((1+\delta)\cdot s - (1-\delta))\cdot(s+1)}.$$

We recover the original systems (7) if $\delta = \frac{1}{17}$. Patel showed in (Patel 1999) that for all complex $\delta \neq 0$ with $|\delta| < \frac{1}{16}$, the three systems (8) are not simultaneously stabilizable by a single time-invariant controller. Thus, especially the question from (Blondel et al. 1994) as stated above has been answered. In turn, Patel put the following question still unanswered: May the three systems given by (8) be simultaneously stabilized for $\delta = 1/16$?

We will show by the theory of subordination that this problem is truely transcendental, and no finite-precision or finite-dimensional controller exists.

The domain of stability is the left half-plane. From (8) we find that with $|\delta| < 1, \delta \neq 0$, the plant P_0 is stable, P_1 is unstable, and the difference is given as

$$P_1 - P_0 = \frac{-2\cdot\delta^2\cdot(s-1)}{(1+\delta)\cdot s - (1-\delta)}.$$

One co-prime factorization $P_1 - P_0 = N/D$ is given by

$$\left(\frac{-2\cdot\delta\cdot(s-1)}{(s+1)}\right) / \left(\frac{(1+\delta)\cdot s - (1-\delta)}{\delta(s+1)}\right). (9)$$

By this choice we have $N = -P_0$.

With $R = (U-D)/N$, a stabilizing controller (if it exists) would be given as

$$C = \frac{R}{1 - R\cdot P_0} = \frac{U-D}{N(1+U-D)}. \quad (10)$$

The non-minimum phase zero of N, i.e. the zero outside the domain of stability is 1. The function value of D is $(\frac{(1+\delta)\cdot 1-(1-\delta)}{(1+1)\delta}) = 1$. The interpolation condition for the unit U is thus given as

$$U(1) \overset{!}{=} D(1) = 1. \quad (11)$$

If C is also simultaneously stabilizing $P_3 \equiv 0$ considering the Bezout relation of the co-prime factors we find that C must be stable. Combining this with the interpolation condition (11) and analysing (10) we find that $1 + U - D$ must be a unit as well as U. The explicit expression for D from (9) yields

$$1 + U - D = U + \frac{\delta\cdot(s+1) - (1+\delta)\cdot s + (1-\delta)}{\delta(s+1)}$$
$$= U - \frac{1}{\delta}\frac{s-1}{s+1}. \quad (12)$$

As the units constitute a group, we may assume $U(s)$ to be of the form $1/U_1$, and multiply the expression (12) by the unit δU_1. The result is again a unit. We find that to solve the problem of simultaneous stabilization of the three plants we will have to produce two units

$$\delta - \frac{s-1}{s+1}U_1(s) \quad \text{and} \quad U_1(s) \quad \text{with} \quad U_1(1) = 1. \quad (13)$$

4.1 Circles of values and unit existence

Suppose we transfer from the time-domain $\Re s < 0$ to the frequency domain. A function is a unit function on the closed unit circle $\overline{\mathbb{D}}$ if and only if it has no zeros and poles there. Thus, the unit interpolation problem in the half-plane (13) translates as follows.

Find a holomorphic function $U(z)$ on $\overline{\mathbb{D}}$ which has no zeros there such that the same holds true for $\delta - zU(z)$, and moreover the interpolation condition $U(0) = 1$ is satisfied.

Obviously, this condition may not be satisfied whenever δ is in the range of values of $zU(z)$ on $\overline{\mathbb{D}}$. Hence, the function $zU(z)$ must omit δ. The following result (cf. (Littlewood 1944)) on covered discs is useful.

Theorem 9. (Hurwitz-Carathéodory) Given a function $f(z)$ holomorphic on $\mathbb{D}$ with the following properties:

$$f(\tilde{z}) = 0,\ \tilde{z} \in \mathbb{D} \Leftrightarrow \tilde{z} = 0, \text{and } f'(0) = 1.$$

Then the function takes every value of the open disc centered at the origin with radius $\frac{1}{16}$. This estimate is sharp.

Suppose $U(z)$ solves our transformed problem. Put

$$f(z) := z \cdot U(z).$$

Obviously, $f(0) = 0$, and $f(z)$ has no other zeros inside $\overline{\mathbb{D}}$ as $U(z)$ is supposed to be a unit. Differentiating, we have $f'(z) = U(z) + z \cdot U'(z)$, and hence $f'(0) = U(0) = 1$ from the interpolation condition. Thus, by Theorem 9 the values of $f(z)$ cover the open disc around the origin of radius $1/16$. Therefore, $\delta - f(z) = \delta - z \cdot U(z)$ will have a zero for every δ with $|\delta| < 1/16$, and can be no unit function. This is Patel's solution to the problem posed by Blondel et al. for the systems (7).

We turn to the proof of Theorem 9 as given in (Littlewood 1944) (for another proof, see (Nehari 1952)). The proof demonstrates subordination of $f(z)$ to the elliptic modular function, and hence additionally allows to give a negative answer to Patel's question for the domain of rational controllers.

Suppose a function $F(z)$ omits the values 0 and β, i.e. for no finite value of z it holds that $F(z) = 0$ or $F(z) = \beta$. Then $F(z)/\beta$ omits 0 and 1. The conformal mapping function of the halfplane to $\mathbb{C} \setminus \{0, 1\}$ is well-known, it is the elliptic modular function (for properties and formulas, see, e.g., (Nehari 1952)). Denote this function by $\lambda = k^2$. This function has a product expansion and a functional equation which employed to the composite mapping from the unit circle $\mathbb{D}$ to the upper-halfplane (via $1/\pi i \cdot \log z$) to $\mathbb{C}$ (via λ) yield

$$\begin{aligned} Q(z) := \lambda(1/\pi i \cdot \log z) &= 16z \cdot \prod_{n=1}^{\infty} (\frac{1 + z^{2n}}{1 + z^{2n-1}})^8, \\ &= 16 \cdot [z - 8z^2 + 44z^3 - 192z^4 + 718z^5 \ldots]. \end{aligned}$$

The essential property of λ implies that $Q(z) \neq 0$ whenever $0 < |z| < 1$. It may be shown that $Q(z)$ is regular at the origin and attins the value zero.

From the theory of subordination, esp. Th. 4, we find putting for $F(z)$ the function $f(z) = zU(z)$ that

$$16 \geq \frac{|F'(0)|}{|\beta|} = \frac{|f'(0)|}{|\beta|}. \qquad (14)$$

The normalization condition $f'(0) = U(0) = 1$ leads to the constant $1/16$ of Theorem 9 above. The higher coefficients of $Q(z)$ yield constraints for the expansion of $U(z)$.

Equality may hold in (14) just for the elliptic modular function by the principle of subordination, esp. Th. 6. In control terms this means that stabilization of the three systems (8) with $\delta = 1/16$ is possible only with the precise infinite controller determined by the inverse of the elliptic modular function. We may hence answer Patel's question in the negative if a controller has to be realized with limited precision or finitely many elements.

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